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# CONTINUOUS NOWHERE DIFFERENTIABLE FUNCTIONS AND ALGEBRAIC INTEGERS

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**1. Introduction.** Examples of continuous, nowhere differentiable functions are well known [**1**] and go back to at least 1860, see [**2**, pp. 954-956] for historical discussion. In most of these examples one considers an infinite series $\sum f_n(x)$, where each $f_n$ is small in some norm, but "wiggles a lot." We present a curious example of another construction, based on some properties of algebraic numbers. The basic idea is as follows. Let $\beta > 1$ be a fixed number, $\beta$ not equal to a rational integer. Every real number $x \in [0,1]$ can be uniquely (in a sense defined below) represented as $x = \sum_{n=1}^{\infty} \varepsilon_n(x)\beta^{-n}$ (representation in base $\beta$). Let now $|\alpha| > 1$ be another number, and define a function

$$f_{\alpha,\beta}(x) = \sum_{n=1}^{\infty} \varepsilon_n(x)\alpha^{-n}. \tag{1}$$

When $\alpha$ and $\beta$ are chosen arbitrarily, the function $f_{\alpha,\beta}$ so defined is generally not continuous. In some special cases, however, when $\alpha$ and $\beta$ are conjugate algebraic integers, the function $f_{\alpha,\beta}$ defined by (1) turns out to be continuous, but nowhere differentiable. We should remark that not every pair of conjugate algebraic integers leads to such a function, but we will give a condition which allows one to construct plenty of examples.

**2. $\beta$-Expansions of real numbers.** We recall here some basic definitions and results, most of which can be found in A. Renyi [**4**] and W. Parry [**3**]. We change the notation slightly and modify some of the results from these two sources in order to suit our specific needs.

Let $\beta > 1$, $\beta$ not equal to a rational integer, be fixed. For every $x$, $0 \le x \le 1$, define a sequence of integer "digits" $\varepsilon_n(x)$ and "remainders"

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