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ON THE STRUCTURE OF SHIFT-INVARIANT SUBSPACES OF $L^2(T^2, \mu)$

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0. Introduction. Suppose that V_1 and V_2 are commuting isometries on a Hilbert space $\mathcal{H}$. In [**3**] and [**6**], conditions are given for $\mathcal{H}$ to have four-fold Wold and Halmos decompositions with respect to V_1 and V_2. These notions are used in [**4, 5**] to characterize invariant subspaces $\mathcal{M}$ of the Hardy space $H^2(\mathbf{T}^2)$ which are generated by an inner function. Such $\mathcal{M}$ are shown to be those invariant subspaces on which V_1 and V_2 doubly commute, where V_j is now multiplication by the coordinate variable z_j, $j = 1, 2$. More generally, [**1**] describes the invariant subspaces of $L^2(\mathbf{T}^2)$ on which these V_1 and V_2 doubly commute. In this article we explore these ideas in the case $\mathcal{M}$ is an invariant subspace of the weighted space $L^2(\mathbf{T}^2, \mu)$.

1. The univariate case. We begin by considering the univariate analogue. This will shed light on the main problem.

Let μ be a finite nonnegative Borel measure on the unit circle $\mathbf{T}$. Define the isometry V on $L^2(\mu)$ by $(Vf)(z) = zf(z)$. A subspace $\mathcal{M}$ of $L^2(\mu)$ is *invariant* (for V) if $V\mathcal{M} \subseteq \mathcal{M}$. Following [**2**], we describe all of the invariant subspaces of $L^2(\mu)$. This will require the Lebesgue decomposition

$$d\mu = 1_\Gamma w \, d\sigma + 1_{\Gamma^c} \, d\lambda$$

where σ is normalized Lebesgue measure on $\mathbf{T}$, w is a weight function, and $0 = \lambda(\Gamma) = \sigma(\Gamma^c)$.

Theorem 1.1. *A subspace $\mathcal{M}$ of $L^2(\mu)$ satisfies the condition $V\mathcal{M} = \mathcal{M}$ if and only if $\mathcal{M} = 1_\Omega L^2(\mu)$ for some Borel set Ω.*

The proof is very similar to that of [**2**, Theorem 2].

Theorem 1.2. *A subspace $\mathcal{M}$ of $L^2(\mu)$ satisfies the condition*

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