ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 2, Spring 1995

A NOTE ON SOME UNCOMPLEMENTED SUBSPACES

ZHIDONG PAN

ABSTRACT. We show that nest algebras are, in general, not complemented as subspaces in the Banach space of all bounded linear operators on a given Hilbert space.

1. All the subspaces in this note are closed subspaces. One of the most useful features of Hilbert spaces is that every subspace in a Hilbert space is complemented. For Banach spaces, the situation is quite different. Let T be the unit circle and m the normalized Lebesgue measure on T. Let $H = L^p(T, m)$ and $K = H^p(T, m)$, $1 \leq p \leq \infty$, be the usual Hardy spaces on the unit circle. It is known that K is complemented in H when $1 < p < \infty$ and not complemented when $p = 1$ or ∞. If we let X be a compact Hausdorff space, $C(X)$ be the set of all continuous functions on X and $A \subseteq C(X)$ a uniform algebra, it is not known whether or not A is always uncomplemented as a subspace of $C(X)$. Glicksberg [**2**], Pelczinsky [**4**] and Sidney [**5**] made some significant progress in this direction, but the general question still remains open. In this note we investigate the same problem for nest algebras, which many believe are a noncommutative analogue of Dirichlet algebras.

Let H be a Hilbert space and (BH) be the set of all bounded linear operators on H. A nest $\mathcal{N}$ is a totally ordered set of orthogonal projections. The corresponding nest algebra is

$$\operatorname{Alg}\mathcal{N} = \{A \in B(H) \mid P^{\perp}AP = 0, \ \forall\, P \in \mathcal{N}\}.$$

If we let $H = L^2(T, m)$, where T denotes the unit circle with normalized Lebesgue measure m, $\{e_n \mid n \in Z\}$ denote the usual orthonormal base for $L^2(T, m)$ (where $e_n(z) = z^n$, $z \in T$, $n \in Z$), P_n denote the orthogonal projection of H onto the subspace $[e_n, e_{n+1}, \dots]$, $n \in Z$, where $[\cdot]$ denotes the closed linear span and $\mathcal{N} = \{P_n\}$, $n \in Z$, then $\operatorname{Alg}\mathcal{N}$ is the set of bounded linear operators with lower triangular

Received by the editors on December 20, 1992.
AMS *Mathematics Subject Classification.* 46.