ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 3, Summer 1995

EIGENVALUE ESTIMATES FOR DEGENERATE PARTIAL DIFFERENTIAL OPERATORS

J. MICHAEL WILSON

1. Introduction. Consider the fourth-order operator $L = \Delta^2 - V$, where Δ is the Laplacian on $\mathbf{R}^d$ and $V \in L^1_{\text{loc}}(\mathbf{R}^d)$ is nonnegative. (For reasons that will be evident, we will assume that $d > 4$.) An integration by parts shows that L will be a nonnegative operator, i.e., have no negative spectrum, if

$$\int_{\mathbf{R}^d} |f|^2 V\, dx \le \int_{\mathbf{R}^d} |\Delta f|^2\, dx \tag{1.1}$$

for all $f \in \mathcal{C}_0^\infty(\mathbf{R}^d)$. The work of Fefferman, Phong and others [**3, 2, 4**] shows that (1.1) is true if V's averages over cubes $Q \subset \mathbf{R}^d$ are suitably small. Specifically, let $p > 1$. Then there is a $\gamma(p, d) > 0$ such that if

$$\ell(Q)^4 \left(\frac{1}{|Q|} \int_Q V^p\, dx \right)^{1/p} \le \gamma(p, d)$$

for all Q ($\ell(Q)$ denotes Q's sidelength) then (1.1) holds. (The L^p norm can be replaced by an Orlicz norm of order $L(\log^{1+\varepsilon} L)$; see [**7**].) Moreover, this condition is close to being necessary, since (1.1) implies trivially that

$$\operatorname{supp}_{Q \subset R^d} \frac{\ell(Q)^4}{|Q|} \int_Q V\, dx \le \gamma' < \infty;$$

all one need do is test (1.1) over translates and dilates of a fixed bump function.

What about an operator that has lower-order cross terms? Let $\mathbf{R}^d = \mathbf{R}^{d_1} \times \mathbf{R}^{d_2}$, and suppose that $L_\delta = \delta(\Delta^2) + \Delta_1 \Delta_2 - V$, where $\delta > 0$ and Δ_i is the Laplacian on $\mathbf{R}^{d_i}$. If $\delta > 1$, then the Δ^2 term

Received by the editors on June 20, 1993.
Research supported under NSF grant DMS 8811775.