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STRICTLY POSITIVE DEFINITE KERNELS ON THE CIRCLE

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ABSTRACT. A sufficient condition is given for the strict positive definiteness and for the strict conditional negative definiteness of a real, continuous radial kernel on the circle. In addition, some necessary conditions are also given, nearly characterizing these kernels.

1. Introduction. On the unit circle S^1, let d_1 be the geodesic distance. The purpose of this paper is to address the problem of finding a continuous function $f : [0, \pi] \to \mathbf{R}$ for which $f \circ d_1$ is either a strictly positive definite or strictly conditionally negative definite kernel. Following [**1**], we say that a function $f : S^1 \times S^1 \to \mathbf{R}$ is a *positive definite kernel* if and only if

$$\sum_{i,j=1}^{n} c_i c_j f(x_i, x_j) \geq 0$$

for all $n \in \mathbf{N}$, $\{x_1, x_2, \ldots, x_n\} \subset S^1$, and $\{c_1, c_2, \ldots, c_n\} \subset \mathbf{R}$. We say that the function f is a *conditionally negative definite kernel* if $f(x, y) = f(y, x)$ for all $x, y \in S^1$ and

$$\sum_{i,j=1}^{n} c_i c_j f(x_i, x_j) \leq 0$$

for all $n \geq 2$, $\{x_1, x_2, \ldots, x_n\} \subset S^1$ and $\{c_1, c_2, \ldots, c_n\} \subset \mathbf{R}$ with $\sum_{j=1}^{n} c_j = 0$. If the above inequalities are strict whenever $x_1, x_2, \ldots, x_n$ are different and at least one of the $c_1, c_2, \ldots, c_n$ does not vanish, we say that the kernel f is *strictly positive* (respectively, *strictly conditionally negative*) *definite.*

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