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ON THE ZETA FUNCTION VALUES $\zeta(2k+1)$, $k = 1, 2, \ldots$

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ABSTRACT. In determinantal form new series representations of the values $\zeta(2k+1) := \sum_{n=1}^{\infty} n^{-2k-1}$, $k = 1, 2, \ldots$, are presented. These follow from a certain trigonometrical identity, which seems to have some independent interest.

1. Introduction. The Riemann zeta function ζ is defined for each complex number s having real part greater than 1 as follows.

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

As intimated in the title we are here concerned about the values $\zeta(s)$ when s is restricted to odd integral values not less than 3. Apery [**1**] helped to rekindle interest in these values when he established the irrationality of $\zeta(3)$. However, for each integer $k > 1$, the arithmetical character of $\zeta(2k+1)$ is entirely unsettled. Several authors have found new series representations for some or all of the values $\zeta(2k+1)$, $k = 1, 2, \ldots$. Ramanujan [**4**] discovered (without proof) that: if α and β are positive real numbers such that $\alpha\beta = \pi^2$ and n is a positive integer, then

$$\begin{aligned}\alpha^{-n}\left\{\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty} \frac{k^{-2n-1}}{e^{2\alpha k} - 1}\right\} &= (-\beta)^{-n}\left\{\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty} \frac{k^{-2n-1}}{e^{2\beta k} - 1}\right\} \\ &\quad - 2^{2n} \sum_{k=0}^{n+1} (-1)^k \frac{B_{2k}}{(2k)!} \frac{B_{2n+2-2k}}{(2n+2-2k)!} \alpha^{n+1-k} \beta^k,\end{aligned}$$

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