ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 3, Summer 1995

A CURIOUS PROPERTY OF THE ELEVENTH FIBONACCI NUMBER

B.M.M. DE WEGER

1. Introduction. As usual we denote by F_n the nth Fibonacci number, defined recursively by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$.

The decimal expansion of the reciprocal of the eleventh Fibonacci number $F_{11} = 89$ has a remarkable shape: its six leading digits are the first 6 terms of the Fibonacci sequence, viz.,

$$\frac{1}{89} = 0.011235955\ldots .$$

Looking more closely, it becomes apparent that the relation goes even beyond the sixth decimal place:

$$\frac{1}{89} = \frac{0}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{5}{10^6} + \frac{8}{10^7} + \frac{13}{10^8} + \frac{21}{10^9} + \frac{34}{10^{10}} + \frac{55}{10^{11}} + \cdots$$

seems even to hold, and it is not difficult to show that, indeed,

$$\frac{1}{89} = \sum_{k=0}^{\infty} \frac{F_k}{10^{k+1}}.$$

This raises the question, posed to me by Ray Steiner, whether a similar phenomenon occurs for expansions in the base y number system of reciprocals of Fibonacci numbers for values of y other than 10. A quick inspection shows that it happens also for $y = 2, 3, 8$, viz.,

$$\frac{1}{F_1} = \frac{1}{F_2} = \frac{1}{1} = \sum_{k=0}^{\infty} \frac{F_k}{2^{k+1}},$$

$$\frac{1}{F_5} = \frac{1}{5} = \sum_{k=0}^{\infty} \frac{F_k}{3^{k+1}},$$

$$\frac{1}{F_{10}} = \frac{1}{55} = \sum_{k=0}^{\infty} \frac{F_k}{8^{k+1}}.$$

Received by the editors on October 25, 1994.