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NORMING SETS AND COMPACTNESS

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ABSTRACT. Let $(X, \| \quad \|)$ be a Banach space and B a norming subset of the closed unit ball B_{X^*} of the dual space X^*. It is proved that if (B_{X^*},weak^*) is sequentially compact then the convex hull of the norm bounded $\sigma(X,B)$-relatively compact subsets of X are $\sigma(X,B)$-relatively compact (Moreover, when (B_{X^*},weak^*) is angelic the norm bounded $\sigma(X,B)$- relatively countably compact subsets of X are $\sigma(X,B)$-relatively compact). As a consequence, if B is assumed to be a boundary of B_{X^*} (i.e. for every $x \in X$ there exists $e^* \in B$ such that $e^*(x) = \| x \|$) then the norm bounded $\sigma(X,B)$- relatively compact subsets of X are relatively weakly compact.

This note addresses the study of some aspects of the compact subsets of Banach spaces X endowed with topologies coarser than their weak topologies. It is well known that for a given Banach space X the classical theorems of Krein-Smulian (about the compactness of the closed convex hull of compact sets), Eberlein-Grothendieck (about the coincidence between relatively countably compact and relatively compact sets) and Eberlein-Smulian (about the coincidence between relatively countably compact, relatively compact and relatively sequentially compact sets) are true for any locally convex topology between the weak and the norm topology of X. Our aim here is to show that, under some general assumptions on the dual unit ball B_{X^*} of X^*, the previous theorems are still true for some topologies in X of the kind $\sigma(X,B)$, where B is any norming subset of B_{X^*}.

Our notation is standard: $(X, \| \quad \|)$ will be a real Banach space, X^* its dual and B_X, respectively B_{X^*}, the unit ball of X, respectively of X^*. A subset B of the dual unit ball B_{X^*} is said to be norming, respectively a boundary for B_{X^*}, if $\| x \| = \sup\{|x^*(x)| : x^* \in B\}$ for

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