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A CHARACTERIZATION OF INNER PRODUCT SPACES BASED ON ORTHOGONAL RELATIONS RELATED TO HEIGHT'S THEOREM

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ABSTRACT. We study two orthogonal relations in a real normed space related to the height's theorem and some characterizations of inner product spaces are obtained.

Orthogonal relations in real normed spaces have been studied with some detail (see [**2**]) in relation with characterizations of inner product spaces and the study of orthogonal additive mappings (see [**1, 6, 7**]). The most classical orthogonal relation in a normed space $(E, \|\ \|)$ is the Pythagorean relation $\perp^P$ defined through Pythagora's theorem:

$$(1) \qquad x \perp^P y \quad \text{if} \quad \|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

Let us note that in inner product spaces Pythagoras theorem is equivalent to the height's theorem: the height over the hypotenuse is the geometric mean of the two divisions of the hypotenuse determined by the foot of the height (i.e., the height divides the triangle into two homothetic pieces). This observation has motivated the formulation of another orthogonal relation $\perp^H$ alternative to (1), i.e., if x and y are the legs, $x-y$ the hypotenuse and $1/(\|x\|^2+\|y\|^2)(\|y\|^2x+\|x\|^2y)$ the foot of the height, we define

$$x \perp^H y \quad \text{if} \quad \left\| \frac{\|y\|^2x + \|x\|^2y}{\|x\|^2+\|y\|^2} \right\| = \left[\left\| \frac{\|y\|^2(x-y)}{\|x\|^2+\|y\|^2} \right\| \cdot \left\| \frac{\|x\|^2(x-y)}{\|x\|^2+\|y\|^2} \right\| \right]^{1/2},$$

or, equivalently,

$$(2) \qquad x \perp^H y \quad \text{if} \quad \|x-y\|\,\|x\|\,\|y\| = \big\|\,\|y\|^2x + \|x\|^2y\big\|.$$

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