ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 4, Fall 1995

ON CLS-MODULES

A. TERCAN

In this note we consider CLS-modules. Let R be a ring with identity, and let M be a right R-module which is the direct sum of its submodules M_1 and M_2. At this case, we show that if M_1 and M_2 are CLS-modules such that M_1 is M_2-injective, then M is a CLS-module. In particular, if M_1 is a CS-module and M_2 is a CLS-module such that M_1 is M_2-injective, then M is a CLS-module.

Throughout this paper all rings will have identities and all modules will be unital. Let R be any ring and M a right R-module. A submodule N of M is called a *complement* (in M) if N has no proper essential extension in M, and the module M is called a CS-*module* provided every complement in M is a direct summand of M (see, for example, [**2, 3, 6, 7**]).

Recall that a direct sum of CS-modules need not be a CS-module (see, for example, [**10**, Example 10]). In [**6**, Theorem 1] Kamal and Muller proved that a module M_R is CS if and only if $M = Z_2(M) \oplus N$ where $Z_2(M)$ and N are CS-modules and $Z_2(M)$ is N-injective. Recently in [**5**, Theorem 8] Harmanci and Smith showed that if $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is a finite direct sum of relatively injective modules M_i, $1 \leq i \leq n$, then M is a CS-module if and only if M_i is a CS-module for each $1 \leq i \leq n$. Kamal and Muller's theorem [**6**, Theorem 1] allows us to consider nonsingular CS-modules. In this paper we define CLS-modules as a generalization of CS-modules, and we think of when the finite direct sums of CLS-modules is a CLS-module.

Let R be a ring and M a right R-module. We will use $Z(M)$ and $Z_2(M)$ to indicate, respectively, the singular submodule of M and the Goldie torsion (second singular) submodule of M.

Definition 1. A submodule N of M is a *closed submodule* of M provided M/N is nonsingular. Note that the concept 'closed submodule' has been used by some other authors. For example,

Received by the editors on September 10, 1993.