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AN ARITHMETIC APPROACH TO THE DAVENPORT-HASSE RELATION OVER $GF(p)$

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ABSTRACT. It is shown how the Davenport-Hasse relation for Gauss sums over $GF(p)$ can be deduced from two simple arithmetic results.

1. Introduction. In this paper we prove two simple arithmetic results and use them to given an elementary proof of the Davenport-Hasse relation (Theorem 3) for Gauss sums over a finite field with p elements, where p is an odd prime. Our first arithmetic result (Theorem 1) gives a congruence $(\mathrm{mod}\, p)$ for a certain root of unity modulo p in terms of factorials. Hudson and Williams [**2**] deduced this congruence from the Davenport-Hasse relation [**1**] and a congruence of Yamamoto [**5**] for Gauss sums. Here we take the reverse approach. We prove Theorem 1 by simple arithmetic manipulations and then use it as a key step in a new proof of the Davenport-Hasse relation; specifically, to determine the root of unity appearing in the relation. The second arithmetic result (Theorem 2) compares the number of integers satisfying two inequalities and is used to establish that the quotient of products of Gauss sums in the Davenport-Hasse relation is an algebraic integer. In addition to these two theorems we need only the basic properties of Gauss sums, Jacobi sums, and the ring of integers of a cyclotomic field. After proving the Davenport-Hasse relation we use it to show that the inequality proved in Theorem 2 is actually an equality.

2. Two arithmetic results. In this section we prove the two results discussed in the introduction.

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