ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 4, Fall 1995

SMOOTH PARTITIONS OF UNITY IN BANACH SPACES

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ABSTRACT. We show that if a Banach space X has an LUR norm, and if every Lipschitz convex function on X can be approximated by C^k-smooth functions, then X admits C^k-smooth partitions of unity, and thus every continuous function on X is a uniform limit of C^k-smooth functions.

1. Introduction and notation. Partitions of unity and smooth approximation on Banach spaces have been studied since the 1950's. For early references on the subject, the reader may refer to the bibliography in [**1**] or [**2**, Chapter 8].

In [**1**], Bonic and Frampton obtained results for classical separable spaces. The nonseparable cases were settled by Toruńczyk [**12**] who used homeomorphic coordinatewise smooth embeddings into spaces $c_0(\Gamma)$. A refinement of this method, in [**4**], was used to extend Bonic and Frampton's results to weakly compactly generated spaces. Building on the idea, McLaughlin [**7**] proved similar results to those we obtain here. In fact, he proved that if a w-LUR norm on a Banach space X can be uniformly approximated on bounded sets by equivalent C^{k+1}-norms, then X admits C^k-smooth partitions of unity.

The more geometrical approach we are following here originates in a paper of Milman [**8**]. It has already provided first-order smoothness results as in Theorem 2.1 in [**14**] (see also [**2**, Theorem 8.3.12]), Theorem 2.2 in [**10**] and in [**11**]. Milman's ideas have also been used in [**9**] to obtain smooth partitions with Lipschitz derivative.

Let us finally mention that our Proposition 2.5 extends Proposition 8.3.10 of [**2**] and provides a result of transfer for smooth partitions of unity.

We recall that a Banach space X admits C^k-smooth partitions of unity if for any open covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of X there exists a family of C^k-smooth functions $\{\Psi_\alpha\}_{\alpha\in\Lambda}$ with the following properties:

Received by the editors on November 20, 1993.