ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 26, Number 2, Spring 1996

QUATERNIONIC BUNDLES ON ALGEBRAIC SPHERES

RICHARD G. SWAN

ABSTRACT. It is shown that for $n \geq 4$ there are nonfree rank 1 algebraic quaternionic vector bundles on the n-sphere which are topologically trivial. For $n \geq 5$ it is shown that there are uncountably many such bundles.

1. Introduction. An old question asks whether there is a bijection between algebraic and topological vector bundles on spheres. More precisely, let $\mathbf{F}$ be one of $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$, and let $VB_k^{\mathbf{F}}(S^n)$ be the set of isomorphism classes of topological $\mathbf{F}$-vector bundles of rank k on the n-sphere S^n. Let $A_n = \mathbf{R}[x_0, \dots, x_n]/(\sum x_i^2 - 1)$ be the coordinate ring of S^n, and let $P_k(\mathbf{F} \otimes_{\mathbf{R}} A_n)$ be the set of isomorphism classes of finitely generated projective $\mathbf{F} \otimes_{\mathbf{R}} A_n$-modules of rank k. The question then is whether $P_k(\mathbf{F} \otimes_{\mathbf{R}} A_n) \to VB_k^{\mathbf{F}}(S^n)$ is a bijection.

The following results are known about this question.

(1) [**16**]. The stable version of the conjecture is true, i.e., $K_0(\mathbf{F} \otimes_{\mathbf{R}} A_n) \to K_{\mathbf{F}}^0(S^n)_{\text{top}}$ is an isomorphism for all n and for $\mathbf{F} = \mathbf{R}, \mathbf{C}$, or $\mathbf{H}$.

(2) [**17**]. The conjecture is true if A_n is replaced by the localization $(A_n)_S$ where $S = \{1 + f_1^2 + \cdots + f_s^2 \mid f_i \in A_n, s \geq 0\}$.

(3) [**15**]. For $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$, it is true for $k \leq 1$ and all n.

(4) [**1**] (see also [**14**]). If $\mathbf{F} = \mathbf{R}$, it is true for $n \leq 2$.

(4) (Murthy, see [**15**]). If $\mathbf{F} = \mathbf{C}$, it is true for $n \leq 3$.

(5) [**15**]. If $\mathbf{F} = \mathbf{H}$ it is true for $n \leq 1$ (and also for $k = 0$ and all n).

Case (6) was observed by the referee of [**15**] who remarked that $\mathbf{H} \otimes_{\mathbf{R}} A_1$ is a principal ideal domain [**12**, Theorem 5.3] (see also Corollary 5.2).

Received by the editors on May 11, 1994, and in revised form on October 10, 1994.
1991 AMS *Mathematics Subject Classification.* Primary 16A50, Secondary 14F05, 55R25.
Key words and phrases. Quaternions, projective modules, algebraic vector bundles.
Partly supported by the NSF.