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ON SOLVABLE QUINTICS $X^5 + aX + b$ AND $X^5 + aX^2 + b$

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ABSTRACT. Let a and b be nonzero rational numbers. We show that there are an infinite number of essentially different, irreducible, solvable, quintic trinomials $X^5 + aX + b$. On the other hand, we show that there are only five essentially different, irreducible, solvable, quintic trinomials X^5+aX^2+b, namely, $X^5 + 5X^2 + 3$, $X^5 + 5X^2 - 15$, $X^5 + 25X^2 + 300$, $X^5 + 100X^2 + 1000$, and $X^5 + 250X^2 + 625$.

1. Introduction. Let Q denote the field of rational numbers, set $Q^* = Q\backslash\{0\}$, and let $f(X)$ be a monic irreducible quintic polynomial in $Q[X]$. If the equation $f(x) = 0$ is solvable by radicals, the quintic polynomial $f(X)$ is said to be solvable. If $f(X)$ is solvable, its Galois group is solvable and is thus contained in the Frobenius group F_{20} of order 20, and hence is isomorphic to F_{20}, D_5 (the dihedral group of order 10) or C_5 (the cyclic group of order 5). It is also known that the discriminant of a solvable quintic is always positive [**1**, p. 390].

Now let $f_i(X) = X^5 + aX^i + b \in Q^*[X]$, $i = 1, 2$, be irreducible and solvable. As disc $(f_i(x)) > 0$, $f_i(X)$ has exactly one real root [**4**, p. 113]. Thus, $f_i(X)$ has nonreal roots and so its Galois group cannot be cyclic and thus must be F_{20} or D_5. For $i = 1, 2$, we define $F(i)$ to be the set of irreducible solvable trinomials $X^5 + aX^i + b$ with Galois group isomorphic to F_{20} and $D(i)$ to be the set of irreducible solvable trinomials $X^5 + aX^i + b$ with Galois group isomorphic to D_5.

We define an equivalence relation on each of $F(i)$ and $D(i)$ as follows: $X^5 + aX^i + b \in F(i)$, or $D(i)$, and $X^5 + a_1X^i + b_1 \in F(i)$, or $D(i)$, are said to be equivalent (written $X^5 + aX^i + b \sim X^5 + a_1X^i + b_1$) if there exists $t \in Q^*$ such that $a_1 = at^{5-i}$, $b_1 = bt^5$, in which case $X^5 + a_1X^i + b_1 = t^5((X/t)^5 + a(X/t)^i + b)$. We denote the set of equivalence classes of $F(i)$ by $\mathcal{F}(i)$ and those of $D(i)$ by $\mathcal{D}(i)$. In Section 2 we prove

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