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ZETA REGULARIZED PRODUCTS AND FUNCTIONAL DETERMINANTS ON SPHERES

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1. Introduction. For $n \geq 2$, let S^n be the n-dimensional sphere with the standard metric, and let Δ_{S^n} be the Laplacian operator on the space of smooth functions. The eigenvalues of this operator are known to be $l(l+n-1)$ with multiplicity β_l^n, where

$$(1) \qquad \beta_l^n = \binom{n+l}{n} - \binom{n+l-2}{n} = \frac{(2l+n-1)}{(n-1)!}\prod_{k=1}^{n-2}(l+k).$$

This paper is concerned with using a factorization theorem for zeta regularized products to compute functional determinants of operators associated with Δ_{S^n}. These determinants are defined by the process of zeta regularization. The most important are $\det'\Delta_{S^n}$ and the determinant of the conformal Laplacian, $\det(\Delta_{S^n} + n(n-2)/4)$. The prime indicates the omission of the zero eigenvalue in the zeta regularized product. In general, the conformal Laplacian is defined to be $\Delta + (n-2)K/4(n-1)$, where Δ is the Laplacian and K is the scalar curvature. For the sphere, $K = n(n-1)$. Computation of the above determinants is equivalent to computing derivatives at $s = 0$ of the zeta function $\sum_{l=1}^{\infty}\beta_l^n[l(l+n-1)]^{-s}$ for the Laplacian and $\sum_{l=0}^{\infty}\beta_l^n[(l+n/2)(l+n/2-1)]^{-s}$ for the conformal Laplacian.

For simplicity, since we are concerned mainly with illustrating the factorization theorem, we restrict our discussion of the conformal Laplacian to the case when n is even. We consider the more general zeta function, $Z_n(s,a) = \sum_{l=1}^{\infty}\beta_l^n[(l+a)(l+n-1-a)]^{-s}$ for integers a, $0 \leq a \leq n-1$, with $a = 0$ corresponding to the Laplacian and $a = n/2$ to the conformal Laplacian. If $a(n-1-a) \neq 0$, then $\det(\Delta_{S^n} + a(n-1-a)) = a(n-1-a)\exp(-Z_n'(0,a))$, and if $a(n-1-a) = 0$, then $\det'\Delta_{S^n} = \exp(-Z_n'(0,a))$.

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