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A SET OF POLYNOMIALS ASSOCIATED WITH THE HIGHER DERIVATIVES OF $y = x^x$

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ABSTRACT. The expansion

$$x^{-x} D_x^n x^x = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1+\log x)^{n-k} F_k(x) \tag{1}$$

is proved, together with the inverse expansion

$$F_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1+\log x)^{n-k} x^{-x} D_x^k x^x. \tag{2}$$

The recurrence $F_n(x) = -D_x F_{n-1}(x) + ((n-1)/x) F_{n-2}(x)$, $n \geq 2$, with $F_0(x) = 1$ and $F_1(x) = 0$ shows that $G_n(x) = x^n F_n(x)$ is a polynomial in x. The fact that $G_n(x) = \sum_{0 \leq j \leq n/2} A_j^n x^j$ with $A_j^{n+1} = (n-j)A_j^n + nA_{j-1}^{n-1}$, $j \geq 1$, where $A_j^n = 0$ for $j < 0$ or for $j > n/2$, shows that $G_n(x)$ is of exact degree $[n/2]$ in x. Finally, in terms of Stirling numbers of the first kind

$$F_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} \sum_{i=0}^{j} \frac{i!}{(i-k+j)!} s(j,i) x^{i-k}. \tag{3}$$

Another curious property is that $\sum_{1 \leq k \leq n} A_k^{n+k} = n^n$, $n \geq 1$. In terms of Comtet-Lehmer numbers,

$$x^n F_n(x) = \sum_{0 \leq j \leq n/2} x^j \sum_{k=n-j}^{n} (-1)^k \binom{n}{k} b(k, k-n+j). \tag{4}$$

An elementary calculus problems asks to find $D_x x^x$. The answer is easily found by logarithmic differentiation and is $x^x(1 + \log x)$. The

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