ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 26, Number 3, Summer 1996

DEPENDENCE OF LOGARITHMS ON COMMUTATIVE ALGEBRAIC GROUPS

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Dedicated to Wolfgang M. Schmidt on the occasion of his sixtieth birthday

ABSTRACT. A well-known conjecture states that linearly independent logarithms of algebraic numbers are algebraically independent over the field of rational numbers. So far, it is not yet known that there exist two algebraically independent logarithms of algebraic numbers. On the other hand, D. Roy has shown that the above conjecture is equivalent to a conjectural description of the rank of matrices whose entries are either algebraic numbers or else logarithms of algebraic numbers. From this point of view, half of the conjecture is known: the actual rank of such a matrix is at least half the conjectural rank.

We consider a similar question for commutative algebraic groups. We show a connection with a density problem, and we prove a partial result by means of the theorem of the algebraic subgroup.

1. The multiplicative group: usual logarithms. Here is the main conjecture for (usual) logarithms of algebraic numbers:

Conjecture 1. *Let $l_1, \dots, l_n$ be complex numbers which are linearly independent over the field $\mathbf{Q}$ of rational numbers. Assume that the n numbers e^{l_i}, $1 \le i \le n$, are algebraic (over $\mathbf{Q}$). Then $l_1, \dots, l_n$ are algebraically independent (over $\mathbf{Q}$).*

As a matter of notations, we shall denote by $\overline{\mathbf{Q}}$ the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$, and by $\mathcal{L}$ the $\mathbf{Q}$-vector space

$$\exp^{-1}(\overline{\mathbf{Q}}^*) = \{z \in \mathbf{C}; e^z \in \overline{\mathbf{Q}}^*\};$$

Received by the editors on December 20, 1994, and in revised form on May 26, 1995.
AMS *Mathematics Subject Classification.* Primary. 11J81, 14L10, Secondary. 11J85, 11J89.