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A NOTE ON HADAMARD ROOTS OF RATIONAL FUNCTIONS

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To Wolfgang Schmidt on the occasion of the celebration of his 60th birthday

ABSTRACT. Suppose F is a polynomial and $\sum_{h\geq 0} F(b_h)X^h$ represents a rational function. If the b_h all belong to a field finitely generated over $\mathbf{Q}$, then it is a generalization of a conjecture of Pisot that there is a sequence (c_h) with $F(c_h) = F(b_h)$ for $h = 0, 1, \ldots$ so that also $\sum_{h\geq 0} c_h X^h$ represents a rational function. We explain the context of this Hadamard root conjecture and make some suggestions that might lead to its proof, emphasizing the apparent difficulties that have to be overcome and the ideas that might be employed to that end.

1. Introduction. Suppose that a polynomial $f(X) \in \mathbf{Z}[X]$ is a cube for all integer values of X. In effect, by the Hilbert irreducibility theorem, but in any case directly, it is easy to show that f is the cube of a polynomial in $\mathbf{Z}[X]$.

In such a spirit, Pisot conjectured, see the remark in [**1**], that if a power sum

$$a(h) = \sum_{i=1}^{m} A_i(h)\alpha_i^h, \qquad h = 0, 1, 2, \ldots$$

is a cube for all h then there is a power sum $b(h) = b_h$ so that $a(h) = b_h^3$ for all h. Here the *roots* α_i are distinct numbers and the coefficients A_i are polynomials, say of respective degrees $n_i - 1$. One says that the power sum has *order* $\sum_{i=1}^m n_i = n$. We should recall that a generalized power sum $a(h) = a_h$ provides the sequence (a_h) of Taylor coefficients of a rational function $r(X)/s(X) = \sum_{h=0}^{\infty} a_h X^h$, with

$$s(X) = \prod_{i=1}^{m}(1 - \alpha_i X)^{n_i} = 1 - s_1 X - \cdots - s_n X^n,$$

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