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# A NOTE ON SIEGEL'S LEMMA

D.W. MASSER

Consider a system of $M$ homogeneous linear equations in $N$ variables with coefficients in a number field $K$. If $N > M$ the system always has nontrivial solutions over $K$. For the purpose of finding "small" solutions there are many results in the literature with the general name of "Siegel's lemma." One of the most precise of these is due to Bombieri and Vaaler [**2**].

The results of Bombieri and Vaaler, stated below, involve the height of the system, the heights of the solutions, and the discriminant of the field $K$. It is well-known that the heights enter into the estimates in an optimal way. Recently it was proved by Roy and Thunder [**9**] that, in general, the discriminant must also be present. Their work covers all possibilities for $M$ and $N$ except $N = M + 1$. The purpose of the present note is to settle the remaining case $N = M+1$. It will turn out that in this case the discriminant enters into the estimates of Bombieri and Vaaler also in an essentially optimal way.

Let us now state precisely the results of [**2**] and [**9**] that are relevant to our discussion. It is slightly more convenient to assume that the linear equations are linearly independent, and at the same time to replace the system by its solution space. Thus, let $V$ be a subspace of $K^N$ (as a vector space over $K$) with dimension $N - M$ strictly between 0 and $N$. The height of $V$ was first defined by Wolfgang Schmidt [**11**], in terms of Grassmann coordinates. We shall use the absolute ("nonlogarithmic") projective height $H'(V)$ with $L^2$ norms at the infinite valuations. This coincides with the definition $H(A)$ of [**2**, p. 15] for any matrix $A$ over $K$ with $M$ rows and $N$ columns defining a system $S$ whose solution space is $V$. It also coincides with the definition $H'(S)$ of [**9**].

For a vector $x$ in $K^N$ we define similarly $H(x)$ as the absolute (nonlogarithmic) projective height using instead the $L^1$ norms at the infinite valuations. This coincides with the definition $h(x)$ of [**2**, p. 15]. We also define $H_1(x)$ as the absolute (nonlogarithmic) affine height; if

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