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SEQUENCES OF FIELDS WITH MANY SOLUTIONS TO THE UNIT EQUATION

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Dedicated to Wolfgang Schmidt on the occasion of his 60th birthday

Let K be a number field of degree δ over the rationals $\mathbf{Q}$ and S a finite set of places of K containing the archimedean places M_∞. Let U_S denote the group of S-units of K and U the group of units. Let $s = \#S$. Then, for $\alpha, \beta \in K^\times$, the (two variable) S-unit equation is

$$\alpha x + \beta y = 1, \qquad x, y \in U_S. \tag{1}$$

In its simplest form, $\alpha = \beta = 1$ and $S = M_\infty$; we call the resulting equation

$$x + y = 1, \qquad x, y \in U, \tag{2}$$

the "unit equation." For a general reference to S-unit equations, see [**4**]. Evertse has shown that the number of solutions to (1) is at most $3 \times 7^{\delta+2s}$ [**3**]. The dependence of the bound on s is interesting. An equivalence relation on S-unit equations is given in [**5**], and it is shown there that for fixed K and S, there are only finitely-many equivalence classes of S-unit equations with more than two solutions. Yet, it is shown in [**6**] that with $K = \mathbf{Q}$, $\alpha = \beta = 1$, (1) can have more than $\exp(Cs^{1/2}/\log(s))$ solutions, for some constant $C > 0$. (A conjecture for the correct dependence on s of the number of solutions to (1) with $K = \mathbf{Q}$ and $\alpha = \beta = 1$ is also given in [**6**].)

On the other hand, it is unknown how the number of solutions to (2) should depend on δ. Nagell has shown that, for any $\delta \geq 5$, there are infinitely many number fields K of degree δ over $\mathbf{Q}$ with at least $6(2\delta-3)$ solutions to (2) [**8**]. (This bound is twice what Nagell stated, but Nagell did not distinguish between the solutions $(\varepsilon_1, \varepsilon_2)$ and $(\varepsilon_2, \varepsilon_1)$ to (2).) There is considerable room between Evertse's and Nagell's bounds; the purpose of this paper is to produce sequences of number fields where

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