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LOCAL ARTINIAN RINGS AND THE FRÖBERG RELATION

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0. Introduction. R will denote a local Artinian ring, $\mathbf{m}$ the unique maximal ideal and k the residue field. The *Poincaré series* is $P_R(t) = \sum_{i\geq 0} \dim(\mathrm{Tor}_i^R(k,k))t^i$ and the *Hilbert series* is $H_R(t) = \sum_{i\geq 0} \dim(\mathbf{m}^i/\mathbf{m}^{i+1})t^i$. $P_R(t)$ is a formal power series while $H_R(t)$ is a polynomial (but not the Hilbert polynomial) since R is Artinian.

We consider the *Fröberg relation*, first studied in [**7**]: $P_R(t) = H_R(-t)^{-1}$. R is a *Fröberg ring* if this relation holds. We are interested in determining when R is Fröberg, particularly in the critical case of $\mathbf{m}^3 = 0$ (cf., [**2**]). The Fröberg relation is a strong property; $P_R(t)$ need not even be rational [**1**]. Our main result is: If $\mathbf{m}^3 = 0$ and $\mathbf{m}\cdot \mathrm{ann}\, x = \mathbf{m}^2$ for all $x \in \mathbf{m}\backslash\mathbf{m}^2$, then R is a Fröberg ring.

This work was motivated by the classification problem for Artinian Witt rings (which are necessarily local) with the unique maximal ideal being the ideal I of even dimensional forms. We briefly review the problem. Let W_F be the class of Witt rings of nonsingular quadratic forms over fields L such that $|L^{\cdot}/L^{\cdot 2}|$ is finite and $-1 \in \sum L^2$. The class W_E of Artinian Witt rings of elementary type is defined inductively. Start with the fundamental Witt rings: $\mathbf{Z}/2\mathbf{Z}$, $\mathbf{Z}/4\mathbf{Z}$ and those of local type (Witt rings of certain P-adic fields, cf. [**13**, Chapter 3]). W_E consists of the Witt rings built from the fundamental Witt rings by a finite sequence of fiber products (over $\mathbf{Z}/2\mathbf{Z}$) and group ring extensions (by finite groups of exponent 2.) Lastly, W_A is the class of Artinian abstract Witt rings defined by Marshall [**13**]. Then $W_E \subset W_F \subset W_A$. It is conjectured that $W_E = W_A$, but neither inclusion is known to be an equality.

We show that if $R \in W_E$, that is, R is of elementary type, then R is a Fröberg ring. This motivated our search for sufficient conditions on R to be Fröberg that could easily be checked for abstract Witt rings.

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