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A LATTICE PROOF OF A MODULAR IDENTITY

RICHARD BLECKSMITH, JOHN BRILLHART AND IRVING GERST

ABSTRACT. We give a lattice rearrangement proof of a six-parameter identity whose terms have the form $x^\alpha T(k_1, l_1) \cdot T(k_2, l_2)$, where $T(k,l) = \sum_{-\infty}^{\infty} x^{kn^2+ln}$. A new balanced Q^2 identity is then established through its use.

1. Introduction. In this paper we give a new proof (Theorem 1) of a variant of a fundamental identity we published earlier in [**5**]. This proof is accomplished by an adroit rearrangement of the indexing lattices in the identity.

The formula in Theorem 1 is also shown (Theorem 3) to be equivalent to the earlier formula and is employed here since its form is convenient for carrying out the rearrangement. The formula is then used in Theorem 4 to prove a new balanced trinomial Q^2 identity (see [**5**] for the meaning of this terminology), where Q is the familiar single-variable quintuple product. The proof itself consists of assigning sets of values to the six parameters in the formula, thereby producing a small family of identities, and then showing that the Q^2 identity is equal to a certain linear combination of these identities.

2. The fundamental identity. Throughout this paper we will use the single-variable function T (cf. [**3**, (2)]) defined by

$$T(k,l) \stackrel{\text{def}}{=} \sum_{-\infty}^{\infty} x^{kn^2+ln} = \prod_{n=1}^{\infty}(1-x^{2kn})(1+x^{2kn-k+l})(1+x^{2kn-k-l}).$$

We call an identity a "T^2 identity" if each of its terms has the form $x^\alpha T(k_1, l_1) T(k_2, l_2)$. We also say that a T^2 identity is "balanced" if the first component pair (k_1, k_2) in each of its terms is the same.

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