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DISCRETE GROUPS AND DISCONTINUOUS ACTIONS

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Introduction. Discontinuous actions of groups play an important role in complex function theory, especially in the study of Riemann surfaces. The study of Kleinian groups, Fuchsian groups, the theory of automorphic forms are all rich areas of mathematics with many deep results. The work of Thurston on 3-manifolds has given additional focus to this very rich field of discontinuous group actions. However, there does not seem to be complete agreement regarding notation and terminology. For instance, the concept of "properly discontinuous" action is defined differently by different authors and these definitions in general are not equivalent. The present paper is a *semi-expository paper* devoted mainly to clarifying various concepts and studying the interrelationships between these concepts.

There are two definitions of "properly discontinuous actions" of a group G on a space X frequently used in literature. One of them requires the existence for each $x \in X$ of an open set U in X with $x \in U$ and $\{g \in G \mid U \cap gU \neq \phi\}$ finite. The other requires that, for any compact set K of X, the set $\{g \in G \mid K \cap gK \neq \phi\}$ be finite. Even when X is a very nice space like a manifold the two definitions are not equivalent. To clarify the difference, we introduce the concept of "strongly properly discontinuous actions" as a generalization of "properly discontinuous actions" (see Definition 5). It turns out that if G acts strongly properly discontinuously on X, then for any compact set $K \subset X$ the set $\{g \in G \mid K \cap gK \neq \phi\}$ is finite (Proposition 7). In general, even when X is a very nice space (for instance a manifold) when G acts properly discontinuously on X, the orbit space X/G need not be Hausdorff. When G acts properly discontinuously on a Hausdorff space X, the orbit space X/G is Hausdorff if and only if the action of G on X is strongly properly discontinuous (Proposition 6). Examples

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