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DANIELL-LOOMIS INTEGRALS

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ABSTRACT. In [**2**] and [**3**] for arbitrary nonnegative linear functionals on functions vector lattices an integral extension of Lebesgue power has been discussed. Here we generalize this extension process, prove convergence theorems using a suitable "local convergence in measure," discuss measurability and give characterizations by equality of upper and lower integrals. Riemann-μ, abstract Riemann-Loomis and Bourbaki integrals are subsumed.

0. Introduction. For a semi-ring Ω of sets from an arbitrary set X and $\mu : \Omega \to [0, \infty[$, only finitely additive, an analogue $R_1(\mu, \overline{\mathbf{R}})$ to the space $L^1(\mu, \overline{\mathbf{R}})$ of Lebesgue-μ-integrable functions was introduced by Loomis [**11**]; this has been extended to Banach space-valued functions by Dunford-Schwartz [**4**], and in more general form in [**6, 7**]. Analogues to the Daniell extension process, but without or with weaker continuity assumptions on the elementary integral, have been treated by Aumann [**1**], Loomis [**11**] and Gould [**5**].

The Daniell-Bourbaki integral extension has been generalized with the integral $\overline{I} : \overline{B} \to \mathbf{R}$ introduced in [**2**], starting with any nonnegative linear functional I on a vector lattice B of real-valued functions on X. If Ω is a δ-ring, μ σ-additive, $I = \int .d\mu$ on $B =$ step functions over Ω, then R_1, L^1 and $\overline{B}$ coincide modulo null functions [**3, 9**].

In Sections 2 and 3 we generalize the extension $I|B \to \overline{I}|\overline{B}$ to $I|B \to J|L$ by "localization," using an appropriate local convergence in measure, which is very useful to obtain convergence theorems in a form analogous to the classical ones (some of which are not true for $\overline{B}$). In Section 4 we give various descriptions of the set L of integrable functions, in particular a Darboux-type characterization on L is proved. Always $R_1 \subset L$ (not true for $\overline{B}$), in general $\overline{B}$ has infinite codimension in L, even modulo null functions.

We recall that the abstract space of integrable functions L is constructed similar to the Daniell L^1 and which coincides with L^1 in the

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