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A NOTE ON ASCOLI'S THEOREM

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ABSTRACT. In this note we study several properties of the space $(G, \mathcal{L}_u)$ where G is a vectorial subspace of continuous functions from a topological space T into a Hausdorff topological vector space F and $\mathcal{L}_u$ is the topology of uniform convergence on the members of a cover $\mathcal{L}$ of T directed by inclusion, when $(G, \mathcal{L}_u)$ satisfies Ascoli's theorem. We give a sufficient condition for the members $\mathcal{L}$ to be functionally bounded, in T, and we apply this result in two ways. First, we prove that, in this case, $(G, \mathcal{L}_u)$ is a topological vector space and, second, if T is a Tychonoff space and F is complete, we prove that the only spaces $(G, \mathcal{L}_u)$ which satisfy Ascoli's theorem are, up to topological isomorphisms, those spaces such that every member of $\mathcal{L}$ is compact. We also obtain an application of this result when the topological vector space F is the usual topological vector space $\mathbf{R}$ of real numbers.

1. Introduction. Let T be a topological space and let $(Y, \mathcal{U})$ be a uniform space. If G is a subset of Y^T, the set of all functions from T to Y, we say that G is pointwise bounded if $\{f(x) : f \in G\}$ is relatively compact in $(Y, \mathcal{U})$ for every $x \in T$ and we say that G is equicontinuous if, for every $x \in T$ and any $V \in \mathcal{U}$, there exists a neighborhood W of the point x such that $(f(x), f(x')) \in V$ whenever $f \in G$ and $x' \in W$. Given a topology $\mathcal{T}$ on a subset $\mathcal{H}$ of Y^T, $\mathcal{T}$ is said to satisfy Ascoli's theorem if a subset K of $\mathcal{H}$ is $\mathcal{T}$-compact if and only if K is $\mathcal{T}$-closed, pointwise bounded and equicontinuous.

Compactness criteria in function spaces, in particular those of Ascoli type, have applications in various fields of mathematics, for instance in functional analysis. The prototype of Ascoli's theorem was proved by Ascoli in [**3**] and independently by Arzelà, who acknowledged Ascoli's priority in [**2**]. This classical theorem of Ascoli-Arzelà was the starting-point for investigations on compactness in function spaces, in particular for spaces of continuous functions.

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