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MARKOV PARTITIONS FOR HYPERBOLIC TORAL AUTOMORPHISMS OF $\mathbf{T}^2$

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ABSTRACT. Using continued fractions, we give a direct and constructive proof for the fact that every matrix in $GL(2, \mathbf{Z})$ whose eigenvalues lie off the unit circle is similar over the integers to a matrix with all nonnegative or all nonpositive entries. This was first proven indirectly by R.F. Williams in 1970 [**8**]. Using this result, we give a constructive proof that there always exists a Markov partition with two connected rectangles for a hyperbolic toral automorphism on the two-dimensional torus.

0. Introduction. Our goal is to construct and study Markov partitions with two connected rectangles for all hyperbolic toral automorphisms on the two-dimensional torus. In their paper, *Similarity of automorphisms of the torus*, [**1**], Adler and Weiss give different constructions for specific cases of these automorphisms. They do not, however, include one for the case when the determinant of the automorphism is positive and the trace is negative. We present this in Section 4. In order to do this, we give a constructive proof in Section 2 that, if $\mathcal{A}$ is a matrix in $GL(2, \mathbf{Z})$ whose eigenvalues lie off the unit circle, then $\mathcal{A}$ is similar over the integers to a matrix with all nonnegative or all nonpositive entries. The proof uses the following fact: given such a matrix, we can consider the convergents of the continued fraction expansion of the slope of the unstable eigenvector as lattice points. Under the map $\mathcal{A}$, they will eventually map to other convergents. We prove this in Section 3. In fact, the desired similarity matrix is given by a consecutive pair of these convergents. Section 1 provides some necessary background. Adler has also continued his work in this area and has different unpublished proofs of the same results.

1. Hyperbolic toral automorphisms. Let $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ be the n-dimensional torus. An automorphism of $\mathbf{T}^n$ is determined by a linear automorphism Φ of $\mathbf{R}^n$ whose matrix has integer entries and

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