

CONVEXITY, SCHUR-CONVEXITY AND BOUNDS
FOR THE GAMMA FUNCTION INVOLVING
THE DIGAMMA FUNCTION

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ABSTRACT. We consider inequalities for the ratio $\Gamma(x + \beta)/\Gamma(x)$, with bounds expressed in terms of the digamma function or its derivatives. We show that this type of inequalities naturally arises from convexity or Schur-convexity of some functions. As a result, we re-derive, generalize or improve several inequalities due to Gautschi, Kershaw and Alzer. Also we present some new inequalities of the type introduced by Gurland.

1. Introduction. Many authors investigated inequalities for the ratio

$$(1) \quad Q(x, \beta) = \frac{\Gamma(x + \beta)}{\Gamma(x)}, \quad x > 0, \beta > 0,$$

see the bibliography in [2]. In this paper we consider the bounds for (1) that involve the digamma function $\Psi = \Gamma'/\Gamma$ or its derivatives.

The first result in this area is due to Gautschi [5]:

$$(2) \quad Q(x, \beta) < \exp(\beta\Psi(x + \beta)), \quad 0 < \beta < 1, x > 0.$$

Kershaw [9] improves and complements Gautschi's bound to

$$(3) \quad \exp(\beta\Psi(x + \beta - 1 + \sqrt{1 - \beta})) < Q(x, \beta) < \exp(\beta\Psi(x + \beta/2)),$$

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