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TORAL ARRANGEMENTS AND HYPERPLANE ARRANGEMENTS

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ABSTRACT. We consider pairs, $(T, \mathcal{A})$, where T is a torus and $\mathcal{A}$ is a finite set of characters of T. Then $d\mathcal{A} = \{\ker(d\chi) \mid \chi \in \mathcal{A}\}$ is a finite set of hyperplanes in the Lie algebra of T. Let $\mathcal{O}_T$ be the coordinate ring of T, and $\mathcal{O}_{T,e}$ the local ring of the identity in T. In analogy with hyperplane arrangements, put $y = \prod_i (\chi_i - 1)$, and consider the set, $D(\mathcal{A})$, of derivations, θ, of $\mathcal{O}_T$ that satisfy $\theta(y) \in y\mathcal{O}_T$. The main results are that the localization of $D(\mathcal{A})$ at the identity of T is a free $\mathcal{O}_{T,e}$-module if and only if $d\mathcal{A}$ is a free hyperplane arrangement, and that if this is the case, then the exponents of $d\mathcal{A}$ can be recovered from $\mathcal{A}$.

1. Introduction. Let k be an algebraically closed field of characteristic zero. In analogy with the definition of a hyperplane arrangement, we will define a toral arrangement to be a pair, $(T, \mathcal{A})$, where T is a torus defined over k and $\mathcal{A} = \{\chi_1, \dots, \chi_s\}$ is a finite set of characters of T. Let $\mathfrak{t}$ be the Lie algebra of T. Then if χ is a character of T, its derivative d_χ is a linear functional on $\mathfrak{t}$. Let $d\mathcal{A}$ be the multiset of hyperplanes $\{\ker(d\chi_1), \dots, \ker(d\chi_s)\}$. Then the distinct hyperplanes in $d\mathcal{A}$ form a central hyperplane arrangement in $\mathfrak{t}$. In this way, a central hyperplane arrangement is canonically associated with each toral arrangement.

Define $y = \prod_{i=1}^s (\chi_i - 1)$ in $\mathcal{O}_T$ and $y' = \prod_{i=1}^s d\chi_i$ in $\mathcal{O}_\mathfrak{t}$. Let $\mathrm{Der}_k(\mathcal{O}_T)$ and $\mathrm{Der}_k(\mathcal{O}_t)$ be the modules of k-linear derivations of $\mathcal{O}_T$ and $\mathcal{O}_\mathfrak{t}$, respectively. Then $D(d\mathcal{A})$ is defined to be $\{\theta \in \mathrm{Der}_k(\mathcal{O}_t) \mid \theta(y') \in y'\mathcal{O}_\mathfrak{t}\}$, and $d\mathcal{A}$ is said to be free if $D(d\mathcal{A})$ is a free $\mathcal{O}_\mathfrak{t}$-module. Terao [**7**, Theorem 2.5] has shown, using the homogeneity of y', that $D(d\mathcal{A})$ is a free $\mathcal{O}_\mathfrak{t}$-module if and only if its localization at $\mathfrak{m}_0$ is a free $\mathcal{O}_{\mathfrak{t},0}$-module. Notice that we have modified the standard definitions slightly, see [**6**], since the linear functions $\{d_{\chi_j} \mid 1 \le j \le s\}$ are not necessarily distinct. It will be shown in Section 2, after Corollary 2.5, that our definition of $D(d\mathcal{A})$ gives the same $\mathcal{O}_\mathfrak{t}$-module as the

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