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# SOME IDENTITIES CONNECTING PARTITION FUNCTIONS TO OTHER NUMBER THEORETIC FUNCTIONS

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**Introduction.** Let $A$ be a subset of $N$, the set of all natural numbers. If $n$ belongs to $N$, let $p_A(n)$ denote the number of partitions of $n$ into parts belonging to $A$. Let $\sigma_A(n)$ denote the sum of the divisors of $n$ that belong to $A$. In particular, if $A = N$, then $p_A(n) = p(n)$, the unrestricted partition function, and $\sigma_A(n) = \sigma(n)$; if $A$ is the set of odd natural numbers, then $p_A(n) = q(n)$, the number of partitions of $n$ into odd parts, and $\sigma_A(n) = \sigma^0(n)$, our notation for the sum of the odd divisors of $n$. Let $q_0(n)$ denote the number of partitions of $n$ into distinct odd parts. Let $E(n) = n(3n-1)/2$. The integers $E(\pm n)$, where $n \geq 0$, are know as the pentagonal numbers. Let $T(n) = n(n+1)/2$. The integers $T(n)$, where $n \geq 0$, are known as the triangular numbers. Consider the following general theorem:

**Theorem X.** *Let $f : A \to N$ be a function such that both*

$$F_A(x) = \prod_{n \in A} (1 - x^n)^{-f(n)/n} = 1 + \sum_{n=1}^{\infty} p_{A,f}(n) x^n$$

*and*

$$G_A(x) = \sum_{n \in A} \frac{f(n)}{n} x^n$$

*converge absolutely and represent analytic functions in the unit disk: $|x| < 1$. Let $p_{A,f}(0) = 1$ and $f_A(k) = \sum\{f(d) : d \mid k, d \in A\}$. Then*

$$(1) \qquad n p_{A,f}(n) = \sum_{k=1}^{n} p_{A,f}(n-k) f_A(k).$$

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