ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 29, Number 1, Spring 1999

# CONSTRAINED CONVERGENCE

ROBERT E. HARTWIG AND PETER ŠEMRL

ABSTRACT. The convergence of $PA^NQ$ is investigated. The results are then used to obtain information about the convergence of constrained Picard iteration $Y_N = PX_N$, where $X_{N+1} = AX_N + B$. In particular, it is shown that, for given $P, A$ and $B$ there exists an initial condition $X_0 = C$ for which $Y_N$ converges, exactly when $R[\vartheta B] \subseteq R[\vartheta(I-A)]$, where $\vartheta = [P^T, A^TP^T, \dots, (A^{m-1})^TP^T]^T$ and $m$ is the degree of the minimal polynomial of $A$.

**1. Introduction.** One of the most basic iterations in matrix theory is the Picard iteration (PI) [**1**]:

$$X_{N+1} = AX_N + B \quad \text{with} \quad X_0 = C, \tag{1.1}$$

where $A$, $B$ and $C$ are constant complex matrices and $A$ is $n \times n$. In practice, however, it may be that only the constrained matrix $Y_N = PX_N$ is "observable." Since the PI iterations admits the exact solution

$$X_N = \bigg[\sum_{i=0}^{N-1} A^i\bigg]B + A^NC, \tag{1.2}$$

we see that

$$Y_N = P\bigg[\sum_{i=0}^{N-1} A^i\bigg]B + PA^NC, \tag{1.3}$$

and, hence, that the convergence of both $S_N = P[\sum_{i=0}^{N-1} A^i]B$ as well as that of $PA^NC$ (as $N \to \infty$) ensures that of $Y_N$. The converse may not be true, however, as seen by taking $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ and $P = I$. On the other hand, if we set $U = B - (I - A)C$, then

$$\begin{aligned} P\bigg[\sum_{i=0}^{N-1} A^i\bigg]U &= P\bigg[\sum_{i=0}^{N-1} A^i\bigg]B - P(I - A^N)C \\ &= PX_N - PC, \end{aligned} \tag{1.4}$$

Received by the editors on May 20, 1997.
AMS (MOS) *Mathematics Subject Classification.* 15A24, 15A09, 56F10.