ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 29, Number 4, Winter 1999

GENERALIZED HÖLDER-LIKE INEQUALITIES

FATHI SAIDI AND RAHMAN YOUNIS

ABSTRACT. Let $n \geq 2$ be a fixed integer, and let M be a one-to-one function. For a real number α, we define

$$R_{\alpha,M} = \Big\{ \mathbf{x} = (x_1, x_2, \dots , x_n) : x_1 > 0, \\ (x_i/x_1) \in \text{Domain}\,(M), i = 2, \dots , n \text{ and} \\ \Big[\alpha - \sum_{i=2}^{n} M(x_i/x_1)\Big] \in \text{Range}\,(M) \Big\}.$$

For $\mathbf{x} \in R_{\alpha,M}$ we define $\Phi_{\alpha,M}(\mathbf{x}) = \mathbf{x_1 M^{-1}} \big[\alpha - \sum_{i=2}^{n} \mathbf{M(x_i/x_1)}\big]$. Several inequalities are presented for $\Phi_{\alpha,M}$. As special cases, these inequalities recover many known "Hölder-like" inequalities.

1. Introduction. Let $n \geq 2$ be a fixed integer, and let $\mathbf{R}$ denote the set of all real numbers. Let $a_i, b_i \in \mathbf{R}$, $i = 1, 2, \dots , n$, be such that $a_1^2 - \sum_{i=2}^{n} a_i^2 \geq 0$ and $b_1^2 - \sum_{i=2}^{n} b_i^2 \geq 0$. Then in [**1**] it was shown that

$$\Big(a_1^2 - \sum_{i=2}^{n} a_i^2\Big)^{1/2} \Big(b_1^2 - \sum_{i=2}^{n} b_i^2\Big)^{1/2} \leq a_1 b_1 - \sum_{i=2}^{n} a_i b_i. \tag{1.1}$$

Inequality (1.1) was generalized by Popoviciu in [**8**] and by Bellman in [**3**] as follows. Let $p > 1$, $(1/p) + (1/q) = 1$, $a_i, b_i \geq 0$, $i = 1, 2, \dots , n$, with $a_1^p - \sum_{i=2}^{n} a_i^p \geq 0$, and $b_1^q - \sum_{i=2}^{n} b_i^q \geq 0$. Then

$$\Big(a_1^p - \sum_{i=2}^{n} a_i^p\Big)^{1/p} \Big(b_1^q - \sum_{i=2}^{n} b_i^q\Big)^{1/q} \leq a_1 b_1 - \sum_{i=2}^{n} a_i b_i. \tag{1.2}$$

This is the "Hölder-like" generalization of (1.1). In [**9**] there is a very simple proof of (1.2) for $p > 1$ and the inverse inequality for $p < 1$ is given. Also, Chapter 5 in [**7**] contains generalizations of (1.2).

Received by the editors on September 15, 1997, and in revised form on February 9, 1998.