ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 4, Winter 2002

ON UNIT SUM NUMBERS OF RATIONAL GROUPS

B. GOLDSMITH, C. MEEHAN AND S.L. WALLUTIS

ABSTRACT. The unit sum numbers of rational groups are investigated: the importance of the prime 2 being an automorphism of the rational group is discussed and other results are achieved by considering the number and distribution of rational primes which are, or are not, automorphisms of the group. Proof is given of the existence of rational groups with unit sum numbers greater than 2 but of finite value.

1. Introduction. The relationship between the groups of units of a unital associative ring and the ring itself has been studied in various forms over a long number of years. Prompted by a question of Fuchs [**6**], there has been special interest in the situation in which the ring is the full endomorphism ring of an abelian group, or more generally a module, and the group of units is then the corresponding automorphism group. Recall the definitions from [**10**]: an associative ring $\mathbf{R}$ is said to have the *n-sum property* (for a positive integer n), if every element of $\mathbf{R}$ can be written as the sum of exactly n units of $\mathbf{R}$. Clearly, if this property holds for an integer n, then it also holds for any integer $k > n$, and so we can make the following definition of the unit sum number of a ring $\mathbf{R}$: $\operatorname{usn}(\mathbf{R}) := \min\{n \mid \mathbf{R} \text{ has the } n\text{-sum property}\}$. If there is an element of $\mathbf{R}$ which is not a sum of units, we set the unit sum number to be ∞, while if every element of $\mathbf{R}$ is a sum of units but $\mathbf{R}$ does not have the n-sum property for any n, we set $\operatorname{usn}(\mathbf{R}) = \omega$. The unit sum number of an abelian group or module is defined to be equal to that of its endomorphism ring. There is a considerable body of literature on this topic, often without using the terminology above. The principal works include [**2**], [**5**], [**9**]–[**11**], [**14**], [**16**]–[**18**].

A notable feature of all previous works is that the only finite values determined for a unit sum number have been 1 and 2, the former corresponding to the trivial situation. The focus of the current work is the problem of calculating unit sum numbers of rational groups, i.e.,

1991 AMS *Mathematics Subject Classification.* Primary 20K30, 20K15.
Received by the editors on July 28, 2001, and in revised form on October 8, 2001.