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ALMOST SKEW-SYMMETRIC MATRICES

J.J. MCDONALD, P.J. PSARRAKOS AND M.J. TSATSOMEROS

ABSTRACT. Almost skew-symmetric matrices are real matrices whose symmetric parts have rank one. Using the notion of the numerical range, we obtain eigenvalue inequalities and a localization of the spectrum of an almost skew-symmetric matrix. We show that almost skew-symmetry is invariant under principal pivot transformation and inversion, and that the symmetric parts of Schur complements in almost skew-symmetric matrices have rank at most one. We also use affine combinations of A and A^t to gain further insight into eigenvalue location and the numerical range of an almost skew-symmetric matrix.

1. Introduction. Let $\mathcal{M}_n(\mathbf{R})$ ($\mathcal{M}_n(\mathbf{C})$) be the algebra of all $n \times n$ real (complex) matrices. In this article, we consider matrices $A \in \mathcal{M}_n(\mathbf{R})$, $n \geq 2$, whose symmetric parts have rank one. This means that the spectrum of the symmetric part of such a matrix A consists of the eigenvalue 0 with multiplicity $n-1$ and a simple nonzero eigenvalue, which will be assumed to be positive for simplicity. We shall then refer to A as an *almost skew-symmetric* matrix.

Tournament matrices and their generalizations [**11, 9**] are closely related to almost skew-symmetric matrices and have provided the motivation for this subject. Indeed, $T \in \mathcal{M}_n(\mathbf{R})$ is a *pseudo-tournament* (i.e., rank $(T+T^t+I) = 1$) if and only if $T+\frac{1}{2}I$ or its negative is almost skew-symmetric. If $T \in \mathcal{M}_n(\mathbf{R})$ is a *hypertournament* (i.e., T has zero diagonal entries and $T+T^t = ww^t - I$ for some nonzero $w \in \mathbf{R}^n$) then $T + \frac{1}{2}I$ is almost skew-symmetric.

The spectra of pseudo-tournaments were initially studied in [**11**]. The spectra of almost skew-symmetric matrices and compact operators were considered in [**4,5**]. In [**4**] it was shown that the real eigenvalues of an almost skew-symmetric matrix satisfy certain interesting inequalities.

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