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# FINITE REFLECTION GROUPS AND LINEAR PRESERVER PROBLEMS

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ABSTRACT. Let $G$ be one of the Coxeter groups $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{D}_n$, or $\mathbf{I}_2(n)$, naturally acting on a Euclidean space $V$, and let $\mathcal{L}(G)$ stand for the set of linear transformations $\phi$ of $\operatorname{End} V$ that satisfy $\phi(G) = G$. It is easy to see that $\mathcal{L}(G)$ contains all transformations of the form $X \mapsto PXQ$, $X \mapsto PX^*Q$, where $P, Q$ belong to the normalizer of $G$ in the orthogonal group and $PQ \in G$. We show that in most cases these transformations exhaust $\mathcal{L}(G)$; the only (rather unexpected) exception is the case $G = \mathbf{B}_n$.

**1. Introduction.** Let $G$ be a finite irreducible Coxeter group naturally acting on a finite dimensional real Euclidean space $V$; see [**2, 4**] for related definitions and terminology. The facial structure of the polytope $\operatorname{conv} G$ (the convex hull of $G$) was recently studied in [**5, 13**]. In the present paper we address the linear symmetries of $\operatorname{conv} G$, the linear transformations of the space $\operatorname{End} V$ of linear operators on $V$ preserving the polytope $\operatorname{conv} G$ or, equivalently, preserving $G$. The problem of describing the set $\mathcal{L}(S)$ of linear transformations of $\operatorname{End} V$ preserving a given set $S \subset \operatorname{End} V$ is an example of linear preserver problems, studied by many researchers, see, e.g., [**14**].

One can find many simple transformations belonging to $\mathcal{L}(G)$, e.g., left and right multiplications by elements of $G$ and the operation $T \mapsto T^*$ of taking the adjoint operator obviously belong to $\mathcal{L}(G)$. In fact, the following result can be readily verified for any subgroup $G$ of the orthogonal group $O(V)$.

**Lemma 1.1.** *Let $P, Q$ belong to the normalizer $N(G)$ of $G$ in the orthogonal group $O(V)$, and assume that $PQ \in G$. Then the transformations $X \mapsto PXQ$ and $X \mapsto PX^*Q$ are in $\mathcal{L}(G)$. These transformations constitute a group.*

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