ON THE NORM OF IDEMPOTENTS
IN C^*-ALGEBRAS

J.J. KOLIHA AND V. RAKOČEVIĆ

ABSTRACT. In this paper we study norms of idempotents in C^*-algebras. Results of Ljance, Vidav, Buckholtz and Wimmer on idempotent operators in Hilbert spaces are considered in the setting of C^*-algebras, and simpler new proofs, based on algebraic and spectral–rather than spatial–arguments, are given. We give an application to projections with respect to a-involutions.

1. Introduction. The paper addresses the twin problem of the existence of an idempotent h in a C^*-algebra $\mathcal{A}$ satisfying $h\mathcal{A} = p\mathcal{A}$ and $(1 - h)\mathcal{A} = q\mathcal{A}$, where p, q are given projections (self-adjoint idempotents) in $\mathcal{A}$, and of the exact value of $\|h\|$ if h exists. We denote such an idempotent h by $\pi(p, q)$.

Ljance [10] showed in 1959 that, for Hilbert space operators, $\|h\| = (1 - \|pq\|^2)^{-1/2}$. In 1964 Vidav [15] found necessary and sufficient conditions for the existence of $\pi(p, q)$, again in the case of Hilbert space operators. Pták [13], apparently unaware of the work of Vidav, and originally also of Ljance, gave in 1984 a solution to both problems, and applied it to extremal operators.

Recently the Hilbert space version of the topic was revisited by Buckholtz [3, 4], Galántai [5], Wimmer [16, 17], and the second author [14]. The first author [8] extended Vidav's results to C^*-algebras.

The purpose of this paper is to consider the existence of $\pi(p, q)$ and Ljance's formula in C^*-algebras, and to give alternative simpler proofs of these theorems. The spectral results on two projections in a C^*-algebra given in Lemma 2.4 hold the key to this simplification. We believe that avoiding spatial arguments in Hilbert spaces in favor of

---