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A NEW INTEGRAL REPRESENTATION OF THE RIEMANN ZETA FUNCTION

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ABSTRACT. The series $\sum_{n=1}^{\infty}(1/n^{l+1})e^{-z^k/n^k}$, k is any positive integer, l is a positive odd number and $l \leq 2k-1$, is studied, and for each pair (k,l), an integral representation of the Riemann zeta function is given. For small pairs, this provides known representations.

1. Introduction. In [**2**], Tennenbaum discussed the series $\sum_{n=1}^{\infty}(1/n^2)e^{-z/n}$ and mainly obtained a proof of the functional equation of the Riemann zeta function. In [**6**] Zhang studied the series $\sum_{n=1}^{\infty}(1/n^2)e^{-z^2/n^2}$ and gave two integral representations and three different proofs of the functional equation of the Riemann zeta function. In [**4**], Wu researched the series $\sum_{n=1}^{\infty}(1/n^{k+1})e^{-z^{2k}/n^{2k}}$ and generalized all results in [**6**]. In [**5**], Wu discussed the series $\sum_{n=1}^{\infty} n^{2t}/(n^{2k}+x^{2k})$ and deduced integral representations for the Riemann zeta function which hold for $\mathrm{Re}\,(s) > 1$. Now in this paper we study the series $\sum_{n=1}^{\infty}(1/n^{l+1})e^{-z^k/n^k}$, where k is any positive integer, l is a positive odd number and $l \leq 2k-1$ and imply a new integral representation for the Riemann zeta function which holds for $-l < \mathrm{Re}\,(s) < 0$ or $\mathrm{Re}\,(s) > 0$, that is, we prove the following theorem

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