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FROBENIUS CLASSES IN ALTERNATING GROUPS

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ABSTRACT. We present a method, based on an old idea of Serre, for completely computing Frobenius classes in alternating groups. We contrast this method with other approaches in examples involving the alternating groups A_3 and A_9. The method can be useful for proper subgroups of alternating groups as well, and we present examples involving the 168-element group $PSL_2(7) = GL_3(2)$ and the Mathieu group M_{24}.

1. Introduction. Let $f \in \mathbf{Z}[x]$ be a monic degree n irreducible polynomial. Let X be its set of complex roots. Let G be the Galois group of f. By definition, G is a subgroup of the symmetric group S_X. A standard fact is that G is contained in the alternating group A_X if and only if the field discriminant d of $\mathbf{Q}[x]/f(x)$ is a square. Equivalently, $G \subseteq A_X$ if and only if the polynomial discriminant $D = dc^2$ of f is a square.

In general, for G a finite group we denote by $G^\natural$ its set of conjugacy classes. For example, the class-set $S_n^\natural$ is naturally identified, via lengths of disjoint cycles, with the set P_n of partitions of n. In this paper, we are concerned with the class-set $A_n^\natural$. For $n \geq 3$, the natural map $A_n^\natural \to S_n^\natural$ is never quite injective, as two classes can be sent to one.

In the situation of the first paragraph, for each prime p not dividing d there is a Frobenius class $\mathrm{Fr}_p \in G^\natural$. Let Fr_p^S be the image of Fr_p under the natural map $G^\natural \to S_n^\natural$. Then, as is very well known, Fr_p^S is just the partition giving the degrees of the irreducible factors of f in the ring $\mathbf{Z}_p[x]$, where $\mathbf{Z}_p$ is the ring of p-adic integers. The computation is easier for p not dividing D, as then it suffices to factor over the finite field $\mathbf{F}_p$, rather than over $\mathbf{Z}_p$.

Now, assuming d is a square, and imposing our conventions in Sections 2 and 3 to remove a single global sign ambiguity, one has a natural map from $G^\natural$ to $A_n^\natural$. Let Fr_p^A be the image of Fr_p in $A_n^\natural$.

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