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ON EXTENSIONS OF SIMPLE REAL GENUS ACTION

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ABSTRACT. May has proved recently [**7**] that if a finite simple group G is generated by two elements of order 2 and s, and acts faithfully on a bordered Klein surface X of least possible genus, then $[\mathrm{Aut}\,(X) : G]$ divides 4 and he asked if $[\mathrm{Aut}\,(X) : G] = 4$ can actually occur. The aim of this note is to give a positive answer to this question. First we give necessary and sufficient conditions for the action of G to be so extendible and then we show that $\mathrm{PSL}\,(2, p)$ satisfy these conditions for arbitrary prime p with $p \equiv \pm 1 \bmod 8$.

1. The *real genus* $\rho(G)$ of a finite group G is the minimum algebraic genus of any compact bordered Klein surface on which G acts faithfully as a group of automorphisms. A *real genus action* of G is an action of G on a bordered Klein surface of algebraic genus $g = \rho(G)$. These notions were introduced by May in [**6**]. In [**7**] May proved that if G is a simple finite group with the real genus action on X and G is generated by two elements of order 2 and s, then G is normal in the group $\mathrm{Aut}\,(X)$ of all automorphisms of X, $[\mathrm{Aut}\,(X) : G]$ divides 4 and finally $\mathrm{Aut}\,(X)$ embeds faithfully in $\mathrm{Aut}\,(G)$. In [**7**] May also posed several open problems. The one he considered the most interesting was whether the case $[\mathrm{Aut}\,(X) : G] = 4$ can actually occur. Here we shall give necessary and sufficient conditions for the action of G to be so extended and then we show that $\mathrm{PSL}\,(2, p)$ for $p \equiv \pm 1 \bmod 8$ satisfies these conditions.

2. We shall use the same approach, notations and terminology as in [**6**] and [**7**]. May remarked that in such exceptional cases $|G| = 3(\rho(G) - 1)$ and $\mathrm{Aut}\,(X)$ must be an M^*-group. So $G = \Delta/\Gamma$, where Γ is a bordered surface NEC group and Δ is an NEC-group with signature $(0; +; [3, 3]; \{(-)\})$, since, by [**2**], these are the only NEC groups with

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