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CHERN-SIMONS FORMS ASSOCIATED TO HOMOGENEOUS PSEUDO-RIEMANNIAN STRUCTURES

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ABSTRACT. Forms of Chern-Simons type associated to homogeneous pseudo-Riemannian structures are considered. The corresponding secondary classes are a measure of the lack of a homogeneous pseudo-Riemannian space to be locally symmetric. Explicit computations are done for some pseudo-Riemannian Lie groups and their compact quotients.

1. Introduction. The characterization by É. Cartan of Riemannian locally symmetric spaces as those Riemannian manifolds whose curvature tensor is parallel was extended by Ambrose and Singer in [**1**]. They proved that a complete simply connected Riemannian manifold is homogeneous if and only if it admits a $(1,2)$ tensor field S satisfying certain equations. If $S=0$ then the manifold is Riemannian symmetric.

The purpose of the present paper is to provide forms of Chern-Simons type for each pseudo-Riemannian manifold (M,g) endowed with a homogeneous pseudo-Riemannian structure S. This construction furnishes odd-dimensional differential forms of degree greater than 1, which are null if $S=0$. Under certain conditions, these forms are closed and define secondary classes. Each of such triples (M,g,S) has thus a number of such differential forms, and roughly speaking (when the corresponding group of real cohomology of the manifold is nonzero), the more nonvanishing classes of that kind a manifold has, the less symmetric it is.

We give several examples of such forms on some Lie groups equipped with left-invariant metrics: The three-dimensional unimodular Lie groups, so having instances of Abelian, nilpotent, solvable and simple Lie groups; and the five-dimensional generalized Heisenberg group $H(1,2)$, which is nilpotent. Further, we consider the corresponding

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