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HARMONIC SHEARS OF ELLIPTIC INTEGRALS

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ABSTRACT. We shear complex elliptic integrals to create univalent harmonic mappings and then use the Weierstrass-Enneper formula to construct embedded minimal surfaces. In one particular case we use this approach to construct a family of slanted Scherk's doubly periodic surfaces whose limit is the helicoid. The corresponding conjugate surfaces form a family of Scherk's singly periodic surfaces that transform into the catenoid. These particular families of surfaces are significant in the study of minimal surfaces, and this method of shearing analytic functions to construct them is a novel approach.

1. Introduction. A harmonic mapping is a complex-valued function $f = u + iv$, for which both u and v are real harmonic. Throughout this paper we will discuss harmonic functions that are univalent and sense-preserving on $\mathbf{D} = \{z : |z| < 1\}$. Such mappings can be written in the form $f = h + \bar{g}$, where h and g are analytic and $|h'(z)| > |g'(z)|$ [**2**]. Constructing mappings with these properties is difficult. However, Clunie and Sheil-Small introduced a shearing method on a certain class of analytic functions for doing so. A few recent papers have used this shearing technique [**4, 15**]. In this paper we apply this shearing method to complex elliptic integrals.

One nice aspect of these univalent harmonic mappings is that they lift to embedded, i.e., nonself-intersecting, minimal surfaces via the Weierstrass-Enneper representation formula. Lifting the harmonic mappings formed by the shearing of complex elliptic integrals results in some interesting minimal surfaces. In particular, the minimal graphs associated with the shearing of a particular elliptic integral of the first kind form a one-parameter family of slanted Scherk's surfaces that range from the canonical Scherk's first, or doubly periodic, surface to the helicoid. Doubly periodic minimal surfaces have been studied and classified by minimal surface theorists [**9, 12, 17**]. Our approach is different in that it considers these doubly periodic surfaces by using geometric function theory techniques and the shearing of analytic

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