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GOING-DOWN IMPLIES GENERALIZED GOING-DOWN

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ABSTRACT. It is proved that if a unital homomorphism f of commutative rings satisfies the going-down property GD, then f also satisfies the infinitistic variant, the generalized going-down property GGD (that is, f exhibits going-down behavior for prime ideal chains of arbitrary cardinality).

1. Introduction. All rings considered below are commutative with identity; all ring homomorphisms are unital. Adapting notation for ring extensions in [**9**, p. 28], we let GU, GD, and LO denote the going-up, going-down and lying-over properties, respectively, for ring homomorphisms. As in [**5**], respectively [**6**], we let GGD, respectively GGU, denote the generalized going-down, respectively generalized going-up, property. Intuitively, GGD, respectively GGU, is the generalization of the usual GD, respectively GU, property when predicated for chains of prime ideals of arbitrary cardinality. Evidently, GGD $\Rightarrow$ GD, respectively GGU $\Rightarrow$ GU, and it is natural to ask if the converse is valid. A partial converse was obtained in [**2**, Remark (a)], respectively [**2**, Theorem], where it was shown that GD $\Rightarrow$ GGD, respectively GU $\Rightarrow$ GGU, for ring extensions $A \subseteq B$ such that each chain of prime ideals of A is well-ordered by reverse inclusion, respectively by inclusion. Subsequently, [**5**] identified many other contexts in which GD $\Rightarrow$ GGD. Additional positive evidence appeared in [**3**, Theorem 2.4], in which the implications GD $\Rightarrow$ GGD and GU $\Rightarrow$ GGU were shown to hold for ring homomorphisms satisfying the strong going-between property SGB (a lifting property for chains that had been introduced by G. Picavet in [**11**]). It was also shown that SGB is equivalent to its infinitistic variant, GSGB [**3**, Corollary 2.3]. Then, at a conference in Venice in June 2002, Kang and Oh announced that GU $\Rightarrow$ GGU for unital ring extensions; it then follows via standard homomorphism theorems that GU $\Rightarrow$ GGU for arbitrary ring homomorphisms. We have benefited from access to the Kang-Oh result [**8**, Corollary 12] in preprint form.

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