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CURVES OF SMALL DEGREE ON CUBIC THREEFOLDS

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ABSTRACT. In this article we consider the spaces $\mathcal{H}^{d,g}(X)$ parametrizing smooth curves of degree d and genus g on a smooth cubic threefold $X \subset \mathbf{P}^4$. For $1 \le d \le 5$, we show that each variety $\mathcal{H}^{d,g}(X)$ is irreducible of dimension $2d$.

1. Introduction. Suppose that $X \subset \mathbf{P}^4$ is a smooth cubic hypersurface in complex projective 4-space. In this article we consider the space $\mathcal{H}^{d,g}(X)$ parametrizing smooth curves of degree d and genus g on a smooth cubic threefold $X \subset \mathbf{P}^4$. For $1 \le d \le 5$ we show that each variety $\mathcal{H}^{d,g}(X)$ is irreducible of dimension $2d$.

For the *Fano scheme* of lines $F = \mathcal{H}^{1,0}(X)$, this is a classical result, cf., [**1**]. We bootstrap from this case by residuation: in each case we show that for a general point $[C] \in \mathcal{H}^{d,g}(X)$ there is a surface $\Sigma \subset \mathbf{P}^4$ which contains C and such that every irreducible component of the residual to C in $\Sigma \cap X$ has degree $e < d$. In this way we inductively prove that for $1 \le d \le 5$ the space $\mathcal{H}^{d,g}(X)$ is irreducible, and in several cases we also show smoothness. In a forthcoming paper [**8**], we use similar methods to describe the *Abel-Jacobi maps* $u_{d,g} : \mathcal{H}^{d,g}(X) \to J(X)$ for $1 \le d \le 5$.

1.1 Notation. All schemes in this paper will be schemes over $\mathbf{C}$. All absolute products will be understood to be fiber products over $\operatorname{Spec}(\mathbf{C})$.

For a projective variety X and a numerical polynomial $P(t)$, $\mathrm{Hilb}_{P(t)}X$ denotes the corresponding Hilbert scheme. For integers d, g, $\mathcal{H}^{d,g}(X) \subset \mathrm{Hilb}_{dt+1-g}X$ denotes the open subscheme parametrizing smooth, connected curves of degree d and genus g.

2. Preliminaries. In this section we gather some preliminary facts about deformation theory, residuation, and Abel-Jacobi maps.

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