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FORMULAS FOR POWERS OF THE HYPERBOLIC TANGENT WITH AN APPLICATION TO HIGHER-ORDER TANGENT NUMBERS

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ABSTRACT. It is shown that the function $\tanh^{2n+1}(x)$ is a linear combination of even-order derivatives of $\tanh(x)$, while the function $1-\tanh^{2n+2}(x)$ is a linear combination of odd-order derivatives of $\tanh(x)$. These results are then used to express higher-order tangent numbers (coefficients in the Maclaurin series for $\tanh^n(x)$) as linear combinations of the ordinary tangent numbers (coefficients in the Maclaurin series for $\tanh(x)$).

1. Introduction. In a recently published book [**2**], the three sequences of polynomials $\{\delta_n\}_0^\infty$ [**2**, Chapter 10], $\{A_n\}_0^\infty$, $\{B_n\}_0^\infty$ [**2**, Chapters 13, 14] were introduced and studied. These sequences are defined by the following recurrences:

$$\delta_{n+2}(x) = x\,\delta_{n+1}(x) + n(n+1)\,\delta_n(x), \tag{1.1}$$

where $\delta_0(x)=1$, $\delta_1(x)=x$,

$$A_{n+2}(z) = \big(z+2(2n+3)^2\big)A_{n+1}(z) - 4(n+1)^2(2n+1)(2n+3)A_n(z), \tag{1.2}$$

where $A_0(z)=1$, $A_1(z)=z+2$, and

$$B_{n+2}(z) = \big(z+8(n+2)^2\big)B_{n+1}(z) - 4(n+1)(n+2)(2n+3)^2B_n(z), \tag{1.3}$$

where $B_0(z)=1$, $B_1(z)=z+8$. The δ_n's are related to the A_n's and B_n's as follows:

$$\delta_{2n+1}(x) = xA_n(x^2), \quad n=0,1,2,\dots, \tag{1.4}$$

and

$$\delta_{2n+2}(x) = x^2B_n(x^2), \quad n=0,1,2,\dots. \tag{1.5}$$

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