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EXPANSIONS OF PRIME IDEALS

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ABSTRACT. When R is an integral domain and S a finitely generated extension of R we investigate the finiteness of the set of prime elements of R that become units in S and the finiteness of the set of prime ideals of R that expand to a given ideal of S. To this end we introduce the notions of GD(1) domains and GD(2) domains. An integral domain is a GD(1), respectively, GD(2), domain if every non-zero element in R is contained in only finitely many principal prime ideals, respectively, prime ideals. We determine when these properties are inherited by subrings, quotient rings, polynomial rings, and power series rings; in this respect GD(1) domains behave like unique factorization domains. A corollary is that if A is an affine (commutative) algebra over a field k, then any field between A and k is algebraic over k. This generalizes the Nullstellensatz. We show that extensions of unique factorization domains studied by Samuel, D.D. Anderson, D.F. Anderson, Zafrullah and many others are proper subclasses of the class of GD(1) domains.

1. Introduction. Let R be an integral domain, and let S be a finitely generated ring extension of R. Thus $S = R[X_1, \dots, X_n]/I$, where I is a constant-free ideal in the polynomial ring $R[X_1, \dots, X_n]$, i.e., $I \cap R = \{0\}$. Our first question is whether the set of (non-associated) prime elements in R that become units in S is finite. For instance, if $R = \mathbf{Z}$, the ring of integers, and $I = \langle p_1 p_2 \dots p_k X - 1\rangle$, where $p_1, \dots, p_k$ are distinct primes, then only these primes become units in $\mathbf{Z}[X]/I$, while $\mathbf{Z}[X_1, \dots, X_n, \dots]/\langle nX_n - 1 : n = 1, 2, \dots\rangle$ is isomorphic to the field of rational numbers, $\mathbf{Q}$.

The canonical embedding of R into S results in a map

$$\phi : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$$

defined by $\phi(P) = P \cap R$ for all proper prime ideals P of S.

By [**12**, Lemma 6D(2)], $\phi(\operatorname{Spec} S)$ is dense in $\operatorname{Spec} R$. When R is a Dedekind domain (so that every nonzero ideal is contained in only a

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