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SUBSPACES WITH NONINVERTIBLE ELEMENTS IN Re $C(X)$

M.H. SHIRDARREH HAGHIGHI

ABSTRACT. Let X be a compact Hausdorff space, and let M be a subspace of $\operatorname{Re} C(X)$ consisting only of noninvertible elements. We show that there exist closed sets $Y \subset X$ such that each element of M has a zero in Y and no closed subset of Y has this property; furthermore, such a Y is a singleton, or has no isolated points. If M has finite codimension n and Y is not a singleton, then Y is a union of at most n nontrivial connected components. We also show that positive functionals exist in $M^{\perp}$.

1. Introduction. Throughout this paper we assume that X is an arbitrary compact Hausdorff space. Denote by $C(X)$, respectively $\operatorname{Re} C(X)$, the space of all continuous complex, respectively real, functions on X.

In this section we discuss the motivation and a brief history of studying subspaces with noninvertible elements in $C(X)$ and $\operatorname{Re} C(X)$.

Plainly, every ideal of $C(X)$ or $\operatorname{Re} C(X)$ is a subspace consisting only of noninvertible elements. Let us call a subspace M of $C(X)$ or $\operatorname{Re} C(X)$ a $\mathcal{Z}$-subspace if M is consisting only of noninvertible elements. In other words, M is a $\mathcal{Z}$-subspace if for each $f \in M$ there exists $x \in X$ such that $f(x) = 0$.

So, every subspace of an ideal in $C(X)$ or $\operatorname{Re} C(X)$ is a $\mathcal{Z}$-subspace. It is easy to construct $\mathcal{Z}$-subspaces in $\operatorname{Re} C[0,1]$ which are not contained in maximal ideals. For example, let $M = \{f : f(0) + f(1) = 0\}$. Each $f \in M$ has a zero in $[0,1]$, by the intermediate value theorem, but clearly M is not contained in an ideal.

The situation for $C(X)$ is completely different. Studying $\mathcal{Z}$-subspaces begins with the following famous result due to Gleason [**2**] and Kahane and Zelazko [**5**]:

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