ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 1, 2006

THE ARC LENGTH OF THE LEMNISCATE $|w^n+c| = 1$

CHUNJIE WANG AND LIZHONG PENG

ABSTRACT. Let $s_n(c)$ be the arc length of the lemniscate $|w^n + c| = 1$, $c \in [0, \infty)$. We obtain some properties of the function $s_n(c)$. In particular, we prove that $s_n(c) \leq s_n(1)$, $c \in [0, \infty)$. We also give the sharp bound for $s_n(1) - 2n$, that is,

$$4 \log 2 < s_n(1) - 2n \leq 2(\pi - 1).$$

1. Introduction. For a polynomial p of degree n, $\{z \in \mathbf{C} \,\big|\, |p(z)| = C\}$ is a curve in the plane known as a lemniscate, where C is a nonnegative constant. Lemniscates have a lot of interesting properties and applications, see, e.g., [**7**]. In 1958 Erdös, Herzog and Piranian proposed the following.

Conjecture A [**3**]. Suppose $p(z)$ is a monic polynomial of degree n, that is,

$$p(z) = \prod_{k=1}^{n} (z - \alpha_k),$$

where $\alpha_k \in \mathbf{C}$, $k = 1, 2, \dots , n$. Write

$$E_n(p) = \big\{z \in \mathbf{C} \,\big|\, |p(z)| = 1\big\}.$$

Then the length $|E_n(p)|$ is maximal when $p(z) = z^n + 1$, which is of length $2n + O(1)$.

This problem has been reposed by Erdös several times, see also [**2**]. Pommerenke obtained many important results on this problem, [**9**–**12**], and gave the first upper estimate [**12**] for the length of $E_n(p)$, namely $|E_n(p)| \leq 74n^2$. In 1995 Borwein [**1**] proved that $|E_n(p)| \leq 8\pi e n$

2000 AMS *Mathematics Subject Classification.* Primary 31A15, Secondary 26D05.
Key words and phrases. Arc length, lemniscate.
This research was supported by NSF of China No. 90104004, 69735020 and 973 project of China G1999075105.
Received by the editors on November 26, 2002, and in revised form on May 8, 2003.