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DIFFERENTIAL INEQUALITIES AND CRITERIA FOR STARLIKE AND UNIVALENT FUNCTIONS

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ABSTRACT. The main aim of this paper is to use the method of differential subordination to obtain a number of sufficient conditions for a normalized analytic function to be univalent or starlike in the unit disc. In particular, we find a condition on β so that each normalized analytic function f satisfying the condition

$$\left|1+\frac{zf''(z)}{2f'(z)}-\frac{zf'(z)}{f(z)}\right|<\beta, \quad z\in\Delta$$

implies that f is univalent or starlike in the unit disc.

1. Introduction. Throughout the text, $\Delta=\{z:|z|<1\}$ denotes the unit disc and $\mathcal{H}$ denotes the class of all analytic functions in Δ. A function $f\in\mathcal{H}$ is said to be convex if $f(\Delta)$ is a convex domain. It is well known that f is convex if and only if $f'(0)\neq 0$ and $\operatorname{Re}\,(zf''(z)/f'(z)+1)>0$ for $z\in\Delta$. A function $f\in\mathcal{H}$ is said to be starlike if f is univalent and $f(\Delta)$ is a starlike domain, with respect to $z=0$. It is well known that f is starlike if and only if $f(0)=0$, $f'(0)\neq 0$ and $\operatorname{Re}\,(zf'(z)/f(z))>0$ for $z\in\Delta$. Let $\mathcal{A}$ be the class of all functions $f\in\mathcal{H}$ such that $f(0)=f'(0)-1=0$. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. In the following, we denote by $\mathcal{K}$ and $\mathcal{S}^*$ the normalized subclasses of functions in $\mathcal{S}$ for which $f(\Delta)$ is convex and starlike, respectively. We denote by $\mathcal{S}^*(\beta)$, the class of all starlike functions f of order β, $\beta<1$, if and only if $f\in\mathcal{A}$ and $\operatorname{Re}\,(zf'(z)/f(z))>\beta$ for $z\in\Delta$. Similarly, f is said to belong to $\mathcal{K}(\beta)$, the class of all convex functions of order β, if and only if $zf'(z)\in\mathcal{S}^*(\beta)$. Note that $\mathcal{S}^*(0)=\mathcal{S}^*$, and $\mathcal{K}(0)=\mathcal{K}$. Define

$$\mathcal{R}=\{f\in\mathcal{A}:\operatorname{Re} f'(z)>0,\ z\in\Delta\}.$$

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