ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 1, 2006

POSITIVE INTEGERS WHOSE EULER FUNCTION IS A POWER OF THEIR KERNEL FUNCTION

J.-M. DE KONINCK, F. LUCA AND A. SANKARANARAYANAN

1. Introduction. For a positive integer n, let $\gamma(n) := \prod_{p|n} p$. The function $\gamma(n)$ is sometimes referred to as either the *algebraic radical* of n, or the *squarefree kernel* of n. Let $\phi(n)$, $\sigma(n)$ and $\omega(n)$ denote the Euler function of n, the sum of the positive divisors of n and the number of distinct prime factors of n, respectively. We also write $P(n)$ for the largest prime factor of n (with the convention that $P(1) = 1$), and $\mu(n)$ for the Möbius function of n.

Jean-Marie De Koninck, see [**3**], asked for all the positive integers n which are solutions of the equation

$$f(n) = \gamma(n)^2, \tag{1}$$

where $f \in \{\phi, \sigma\}$. With $f = \phi$, the above equation has precisely six solutions, and all these are listed in the last section of this paper. With $f = \sigma$, it is conjectured that $n = 1, 1782$ are the only solutions of the above equation, but we do not even know if this equation admits finitely many or infinitely many solutions n. In [**4**], it is shown, among other things, that every positive integer n satisfying equation (1) with $f = \sigma$ can be bounded above by a function depending on $\omega(n)$. In particular, if one puts an upper bound on the number of distinct prime factors of the positive integer n satisfying equation (1) with $f = \sigma$, then one can bound the positive integer n.

In this paper, we let k be any positive integer, and we let E_k be the set of positive integer solutions n for the equation

$$\phi(n) = \gamma(n)^k. \tag{2}$$

We also set $N_k := |E_k|$. It is easy to see that $E_1 = \{1, 2^2, 2 \cdot 3^2\}$. Moreover, for $k \geq 2$, each one of the numbers $1,\ 2^{k+1},\ 2^k \cdot 3^{k+1},\ 2^{k-1} \cdot$

Received by the editors on April 14, 2003, and in revised form on January 16, 2004.