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LINEAR PRESERVERS FOR SYLVESTER AND FROBENIUS BOUNDS ON MATRIX RANK

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ABSTRACT. Let A and B be $n \times n$ matrices. A classical result about the rank function is Sylvester's inequality which states that the rank of the product of AB is at most $\min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$ and at least $\operatorname{rank}(A) + \operatorname{rank}(B) - n$. A generalization of Sylvester's inequality is Frobenius's inequality which states that

$$\operatorname{rank}(AB) + \operatorname{rank}(BC) \leq \operatorname{rank}(ABC) + \operatorname{rank}(B).$$

In this paper we investigate the structure of linear operators that preserve those ordered pairs or triples of matrices which satisfy one of the extreme cases in these inequalities.

1. Introduction. Let $\mathbf{F}$ be any field, and let $\mathcal{M}_n(\mathbf{F})$ denote the space of all $n \times n$ matrices with entries from $\mathbf{F}$. Let $\rho(A)$ denote the rank of A. Let $E_{i,j}$ be the matrix with a "1" in the (i,j) position and zero elsewhere.

Definition 1.1. If $T : \mathcal{M}_n(\mathbf{F}) \to \mathcal{M}_n(\mathbf{F})$ is a linear operator, we say that T is a (U,V)-*operator* provided there exist nonsingular matrices $U, V \in \mathcal{M}_n(\mathbf{F})$ such that either

1. $T(X) = UXV$ for all $X \in \mathcal{M}_n(\mathbf{F})$ or
2. $T(X) = UX^tV$ for all $X \in \mathcal{M}_n(\mathbf{F})$,

where X^t denotes the transpose of X.

Note that it follows that T is a (U,V)-operator if and only if T is a composition of operators of type 1 above and the transpose operator.

Some classical inequalities concerning the rank of sums and products are:

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