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DIOPHANTINE APPROXIMATIONS AND A PROBLEM FROM THE 1988 IMO

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ABSTRACT. Harborth has recently shown how to describe all integer solutions to a Diophantine equation arising from a problem at the 1988 International Mathematical Olympiad. Harborth uses a clever reduction method, although it seems that this method is somewhat ad hoc. The purpose of the present paper is to show how the result of Harborth can be proved, and extended, using the classical theory of continued fractions. More to the point, a shortcoming in this classical theory is circumvented by an extension to Legendre's theorem concerning a sufficient condition for a rational integer to be a convergent to a given irrational number.

1. Introduction. Problem 6 from the 1988 International Mathematical Olympiad asked the participants to prove that if a and b are positive integers for which

$$k = \frac{a^2 + b^2}{ab + 1} \tag{1.1}$$

is an integer, then k must be the square of an integer. Recently, Harborth [**3**] has taken this problem one step further by providing closed formulas for the complete solution set of triples of integers (a, b, k) satisfying (1.1).

A simple rearrangement of (1.1), together with the substitution $x = bk - 2a, y = b$ shows that solutions (a, b, k) of (1.1) are in correspondence with solutions (x, y, k) of the quadratic equation

$$x^2 - (k^2 - 4)y^2 = 4k. \tag{1.2}$$

If k is even, or if $\gcd(x, y) > 2$, then it is relatively simple to determine the solutions to (1.2) using the basic theory of continued fractions. To see this, assume first that k is even, $k = 2l$ say, and assume that $k > 2$,

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